Constant-Fractional-Lag Model for Axisymmetric Two-Phase Flow

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The suitability of the constant-fractional-lag model for axisymmetric two-phase flow with small particle loading is examined for an inviscid incompressible counterflow. A counterflow is a low-order approximation for the flow within a solid-rocket motor with a long bore of constant radius. In the model, each component of the particle-phase velocity is expressed as a certain constant multiple of the corresponding component of the gas-phase velocity. A different lag constant is required for the radial and the axial components of the particle-velocity field. For light particle loading, the constant-fractional-lag model yields mathematically accurate solutions (of the formulation) for both small and finite values of the interphase-velocity-slip parameter. Comparisons with results from the Lagrangian-particle-tracking method show excellent agreement at sites outside the Stokes layer holding in that portion of the two-phase flow immediately contiguous to the gas-grain interface; i.e., the agreement holds independently of the initial particle velocity at the solid-gas interface. The constant-fractional-lag model is easier to apply than the Lagrangian-particle-tracking method, and results are conveniently obtained in Eulerian form. Since the validity of the model here is proven only for an inviscid incompressible counterflow, it is recommended that further study be undertaken to delineate the validity of the approximation for more general categories of axisymmetric flow.

I. Introduction

THE effect of condensed products, explicitly, metal-oxide particles, on the performance of solid-rocket motors has been examined theoretically for about a half century.[1,2] Crowe[3] has cited the helpful introduction[4] of the method of characteristics for two-dimensional supersonic two-phase, gas-solid nozzle flow (in connection with the burning of metallized grains); for supersonic nozzle and exit cone flow, computer programs based on this approach are now standard in the industry.[5] It has also been suggested[4] that adoption of a constant interphase slip velocity, and a distinct but still constant interphase slip with respect to temperature, is a highly convenient approximation for one-dimensional flows; such flows are usefully approximated as unidirectional and spatially variant only in that direction. However, though some of the conclusions obtained under the simplification seem plausible,[6] the justification for such a constant-fractional-lag approximation has been questioned.[7] Explicitly, the constant-fractional-lag model takes each component of the particle velocity to be a constant multiple of the corresponding component of the gas velocity. The value of the constant varies from component to component and is a function of the slip parameter. More generally, the value of the constant may vary in space, but its variation must be slow.

Here, we investigate the conditions under which the constant-fractional-lag approximation for the particle-velocity field is justified, and we evaluate possible extensions of the approximation, for an axisymmetric flow pertinent to a long-bore solid-rocket motor[8] with a metallized-composite grain (Figs. 1), such as might be used for a first or second stage. If we adopt quasisteady inviscid incompressible irrotational flow as a simple lowest-order but adequate description of the in-motor gas dynamics, then by inspection[9] (or, more formally, by solution of Laplace's equation subject to Neumann boundary conditions) the in-motor gas-phase velocity components conveniently are those describing a counterflow. That is, in cylindrical polar coordinates, the axial gas-phase velocity component varies linearly with the axial coordinate, and the radial gas-phase velocity component varies linearly with the (cylindrical) radial coordinate.

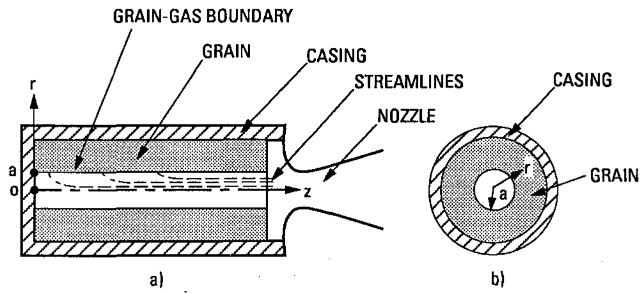

Fig. 1 Schematic of a rocket motor with a simple, constant-bore grain of radius $r = a$: a) side view (counterflow streamlines roughly sketched); b) cross-sectional view at constant axial position z.

Even with the assumption of inviscid irrotational incompressible flow, solution of the continuity and the momentum equations for the particle and gas phases is still complicated. Two useful approximations that have been proposed previously[10] are the small-slip approximation and the light-particle-density approximation. The small-slip approximation is justified if the particle-velocity-equilibration time is small compared with the time characterizing the flowfield. In such circumstances, the particle and gas velocities and densities can be obtained via asymptotic expansions (in terms of a small-slip parameter) and the method of characteristics. Because of the singular nature of the asymptotic expansion, solutions are valid only outside a thin layer near the gas-grain interface. The small-slip approximation does not require the particle density be small compared with the gas density.

On the other hand, if the particle mass density is small compared with the gas density, then, from the momentum equations for the gas phase, it can be shown that the force due to the gas-particle interaction is much smaller than the

Received March 23, 1990; revision received Aug. 1, 1990; accepted for publication Aug. 3, 1990.

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pressure force of the gas. In such circumstances, the gas flowfield is not altered by the presence of the particles to leading order. The interaction force due to the gas-particle interaction does alter the particle trajectories. This approximation does not require that the particle-velocity-equilibration time be small compared with the characteristic flow time.

Subsequent analysis will be based on the light-particle-density approximation. Section II analyzes particle trajectories using the Lagrangian formulation. It will be shown that, for particles not too close to the wall, there is nearly a constant lag between each component of the particle and the gas velocity. The value of the lag constant, different for different components of the velocity vector, is a function of the slip parameter. Comparisons will be made with the Eulerian formulation for conditions in which the particle-velocity-equilibration time is short compared with the time scale of the flowfield. It will be shown that the Eulerian solution based on singular-perturbation expansions agrees with the Lagrangian formulation only outside a thin (Stokes) layer near the wall. The thickness of the Stokes layer depends on the ratio of the particle-velocity-equilibration time to the flow time. Section III analyzes the particle phase using the Eulerian formulation. The constant-fractional-lag model for one dimension[4,11] will be extended to axisymmetric flow, and the validity and the implications of the model will be discussed.

In what follows, we shall take the solid (or molten) particle to be formed instantaneously at the gas-grain interface, which has the configuration of the lateral surface of a right circular cylinder; whereas, in fact, the particles are formed in the near vicinity of the surface, and complete conversion of the metal additive to oxide may be deferred over the particle trajectory. The inclusion of such finite-rate evolution would introduce complication that is not essential to the topic of concern. Also, we shall neglect the role of gravitational acceleration and of particle-particle collisions, effects that appear to be often not important under practical conditions. Finally, we take the particles to be spherical and of such a (given) size that a linear (Stokes-type) drag law[6] applies. The Stokes drag law is adequate for present purposes as long as the Reynolds number based on the velocity difference between the gas phase and the particle phase is of order unity or less. For parameter values of practical interest here, outside the Stokes layers, this drag law suffices.

II. Lagrangian Formulation of the Particle Phase

For light-particle-loading conditions, in which the gas-phase flowfield is taken to be unaffected by the presence of the particles, we adopt a simple axisymmetric counterflow for a constant-bore-radius motor:

$$u_{gr} = -\frac{1}{2}kr \tag{1a}$$

$$u_{gz} = kz \tag{1b}$$

where g indicates the gas phase, and r and z indicate the radial and axial components of the velocity vector, respectively. The azimuthal θ component of the gas velocity is zero. The constant k is inversely related to the time scale characterizing the flow. The gas flowfield described by Eqs. (1a) and (1b) may be confirmed to be irrotational and incompressible. The value of the parameter k is given by $(\rho_1\dot{R})/(\rho_g a)$, where ρ_1 is the density of the portion of the solid grain that evolves to gas, $\dot{R}$ the magnitude of the rate of regression of the grain-gas interface, a the (cylindrical) radial distance to that interface, and ρ_g the density of the gas phase (approximated as spatially uniform within the motor cavity). For plausible values for conditions holding soon after ignition ($\rho_g \approx 4000$ kg/m^3, $\dot{R} \approx 10^{-2}$ m/s, $\rho_g \approx 9.3$ kg/m^3, and $a \approx 0.13$ m), $k \approx 5.6$ s^{-1}. Hence, the axial speed would achieve one-tenth of the value of the speed of sound at a distance over 20 meters downwind from the nose; typically, this is over three times the length of the grain.

The Lagrangian formulation relates the particle acceleration to the Stokes force between the particle and the gas phases through Newton's second law of motion:

$$m_p(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -Cr\dot{\theta} \tag{2a}$$

$$m_p(\ddot{r} - r\dot{\theta}^2) = C\left(-\frac{1}{2}kr - \dot{r}\right) \tag{2b}$$

$$m_p\ddot{z} = C(kz - \dot{z}) \tag{2c}$$

where m_p is the mass per (spherical) particle (typically 10^{-11} kg for a 10-μm-radius particle); C the Stokes constant, $6\pi\rho_g\nu_g r_o$ (where r_o is the particle radius and ν_g is the dynamic viscosity of the gas, typically 40 m^2/s in the motor, so $C \approx 0.074$ kg/s for a 10-μm-radius particle); and (r, θ, z) the coordinate of the particle. The superscript dot denotes the time derivative. In the absence of virtually any reliable experimental data concerning the turbulence (and, for that matter, most aspects) of the in-motor cavity flow, we make no attempt to account for turbulent effects in calculating the particle trajectory. Any speculative formulation of such effects seems as likely to incorporate error as insight, but almost certainly increases the mathematical labor.

The velocity-equilibration time for the particle is defined as

$$\tau_v = m_p/C \tag{3}$$

The slip parameter is defined as the ratio of the particle-equilibration time to the flow time. In our case, it is $k\tau_v$. For $k\tau_v \ll 1$, the small-slip approximation is expected to hold. Here, $k\tau_v \approx 1.6 \times 10^{-5}$ for $r_o = 10$ μm, for parameter values assigned earlier.

We take the radius of a long-bore solid-rocket motor to be constant. The particles are generated at time $t = 0$ at the grain-gas interface. Equation (2a) can be integrated to obtain $[r(0) = a]$,

$$r^2\dot{\theta} = a^2\dot{\theta}(t = 0)\exp(-t/\tau_v) \tag{4}$$

It is clear from Eq. (4) that for $r \neq 0$, i.e., for particles not along the axis of the motor, $\dot{\theta} = 0$ for all time if $\dot{\theta} = 0$ initially. We take this to be the case. The particle flowfield is also axisymmetric. Equations (2) can be simplified to

$$\tau_v\ddot{r} + \dot{r} + \frac{1}{2}kr = 0 \tag{5a}$$

$$\tau_v\ddot{z} + \dot{z} - kz = 0 \tag{5b}$$

subject to initial conditions $r(0) = a$, $z(0) = z_o$.

We take the particle (initial) velocity at the solid-gas interface to be zero. In other words, the particles are dragged into the flowfield by the gas, and so Eqs. (5a) and (5b) are subject to the initial conditions $\dot{r}(0) = 0$ and $\dot{z}(0) = 0$. Equation (5b) can be solved to obtain

$$z(t) = \frac{z_o}{2(1 + 4k\tau_v)^{1/2}}\Bigg([1 + (1 + 4k\tau_v)^{1/2}] \cdot \exp\left\{\frac{1}{2}[-1 + (1 + 4k\tau_v)^{1/2}]t/\tau_v\right\} - [1 - (1 + 4k\tau_v)^{1/2}] \cdot \exp\left\{\frac{1}{2}[-1 - (1 + 4k\tau_v)^{1/2}]t/\tau_v\right\}\Bigg) \tag{6a}$$

The solution of Eq. (5a) depends on the value of $k\tau_v$. For $k\tau_v < 0.5$, the solution of Eq. (5a) is given by

$$r(t) = \frac{a}{2(1-2k\tau_v)^{1/2}}\Bigg([1+(1-2k\tau_v)^{1/2})]$$

$$\cdot \exp\left\{\frac{1}{2}[-1+(1-2k\tau_v)^{1/2}]t/\tau_v\right\}$$

$$-[1-(1-2k\tau_v)^{1/2}]$$

$$\cdot \exp\left\{\frac{1}{2}[-1-(1-2k\tau_v)^{1/2}]t/\tau_v\right\}\Bigg) \tag{6b}$$

For $k\tau_v = 0.5$, the solution is given by

$$r(t) = a\left(1+\frac{1}{2}\frac{t}{\tau_v}\right)\exp[-t/(2\tau_v)] \tag{6c}$$

It is straightforward to show that, for solutions given by Eqs. (6b) and (6c), particles cannot reach the axis of symmetry in finite time. However, for $k\tau_v > 0.5$, the solution of Eq. (5a) contains an exponential function and a sinusoidal function, and particles can reach the axis in finite time. Very near the axis, particle-particle collisions not included in the formulation are anticipated to occur. Such collisions may result in agglomeration. In any case, in a counterflow, such particles seem likely to remain near the axis as they are transported downwind to the nozzle entrance. To simplify the subsequent analysis, we limit attention to cases in which $k\tau_v \leq 0.5$.

Figure 2 shows the r-component velocity difference (normalized by the absolute value of the r-component gas velocity) between the particle phase and the gas phase as a function of time for three different values of the slip parameter $k\tau_v$. The horizontal axis in Fig. 2 is the dimensionless time t', in which the dimensional time t is normalized by the particle-equilibration time τ_v; i.e., $t' = t/\tau_v$. The normalized velocity difference is a maximum at the wall since the particle velocity is taken to be zero at the wall. The radial particle velocity eventually exceeds the radial gas velocity (in absolute value) for $t' = 4$, i.e., for $t = 4\tau_v$. The increase of the particle radial velocity (in absolute value) is due to the interaction between the two phases. Figure 3 shows the z-component velocity difference (normalized by the z-component gas velocity) between the two phases as a function of time for three different values of the slip parameter $k\tau_v$. The axial (z-component) particle velocity is always smaller than the axial gas velocity. From Figs. 2 and 3, it is clear that after three particle-equilibration times have elapsed, each component of the normalized velocity difference is almost a constant, invariant with time. The value of the constant appropriate for each coordinate direction is a function of the slip parameter. This result indicates the usefulness of the constant-fractional-lag model for this particular axisymmetric two-phase flow.

Figures 4 and 5 show the r-component trajectories for the particles and the gas elements, respectively. Away from the axis ($t' \leq 8$), the gas elements are closer to the axis than the particles since the magnitude of the r-component gas velocity is larger than that of the particle velocity away from the axis. For $t' > 8$, particles are closer to the axis than the gas, consistent with the result that the magnitude of the r-component particle velocity is larger than the corresponding component of gas velocity near the axis. Figures 6 and 7 show the z-component particle trajectory and gas element trajectory, respectively. Particles lag behind the gas in the axial direction, consistent with the axial-velocity difference shown in Fig. 3.

It is interesting to compare the particle flowfield obtained from the Lagrangian formulation with that obtained from an Eulerian formulation. In general, solutions in the Eulerian

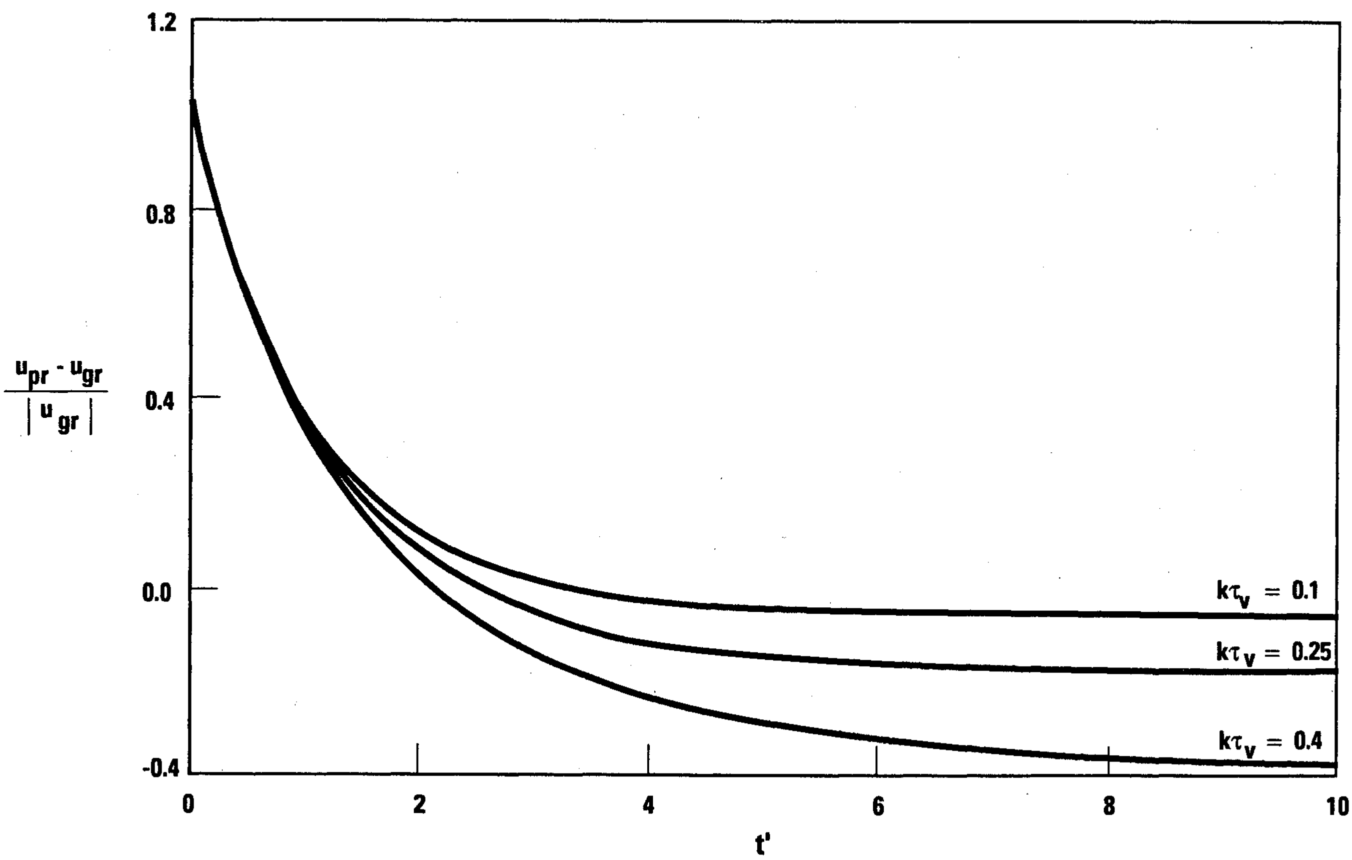

Fig. 2 r-component velocity difference (normalized by the r-component gas velocity) between the particle phase and the gas phase, as a function of time.

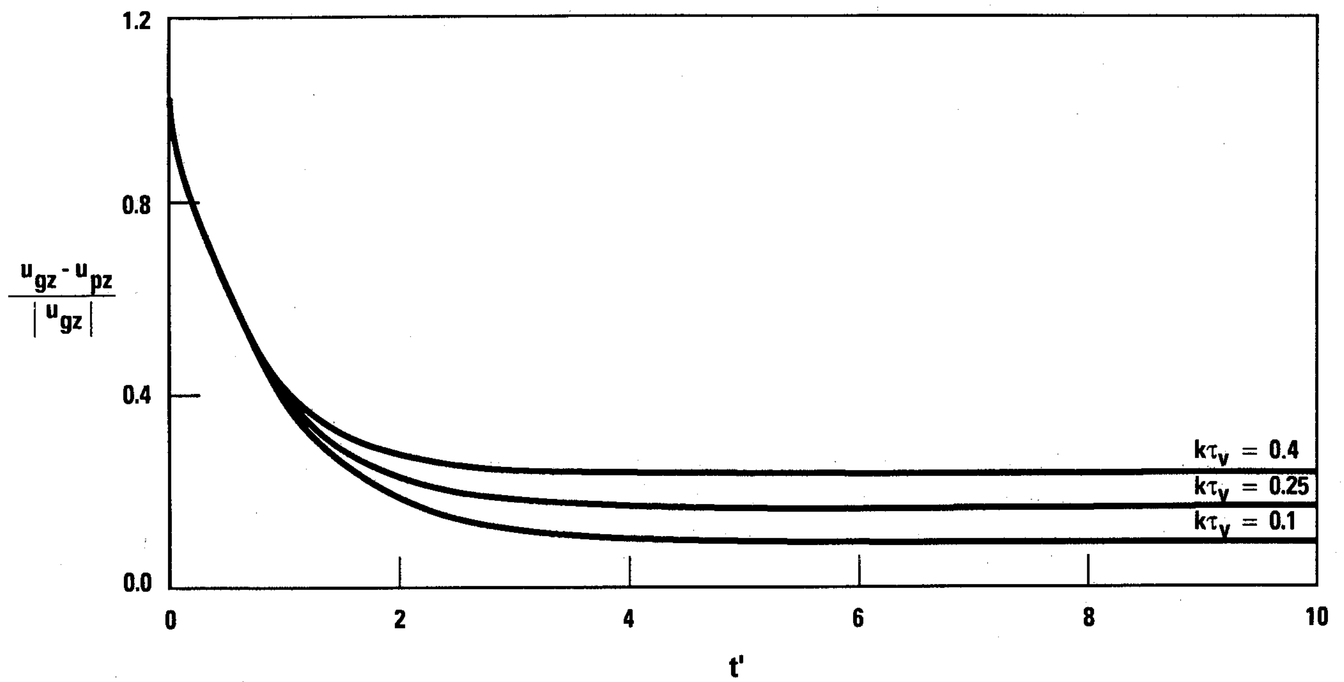

Fig. 3 *z*-component velocity difference (normalized by the *z*-component gas velocity) between the particle phase and the gas phase, as a function of time.

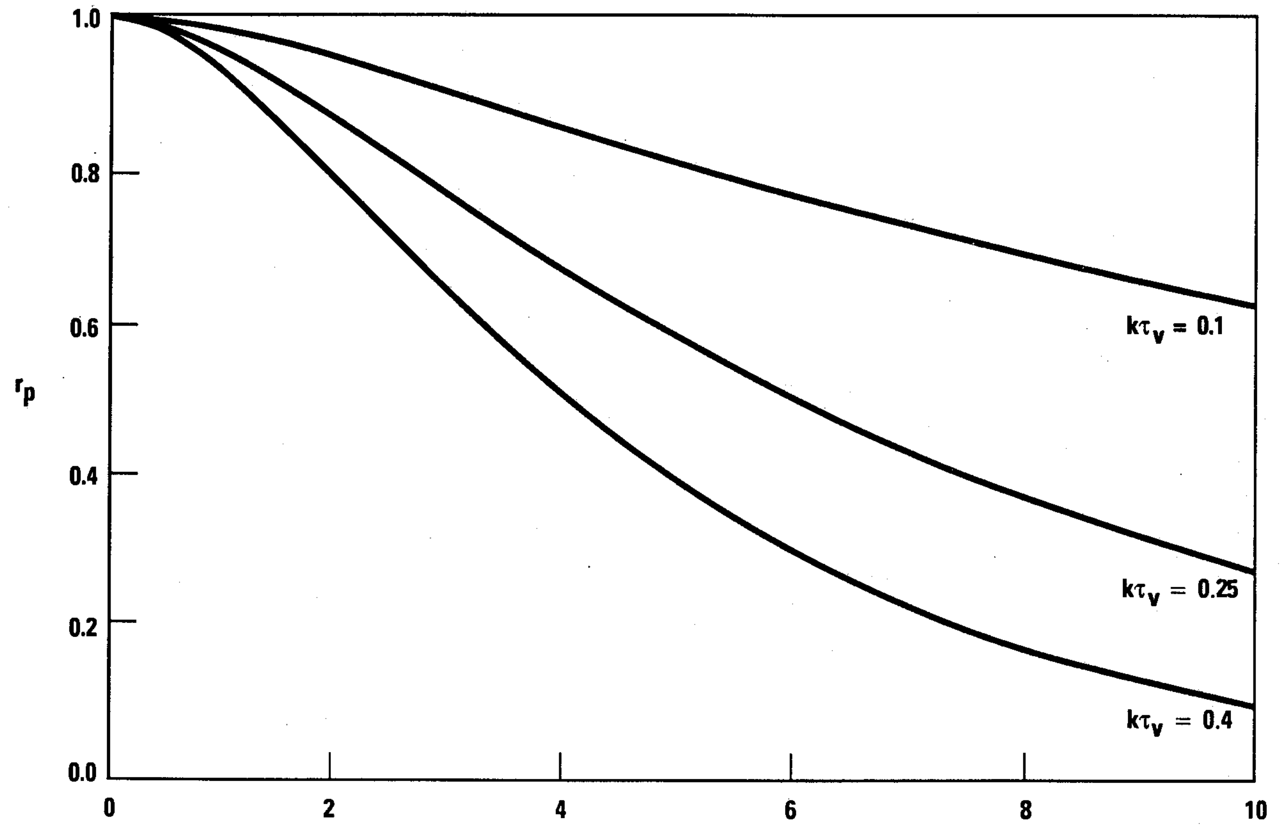

Fig. 4 *r*-component particle trajectory as a function of time.

formulation are difficult to obtain due to the nonlinearity of the momentum equations for the particle flowfield; however, if the slip parameter $k\tau_v$ is small, solutions can be obtained via asymptotic expansions. Let

$$u_{pr} = u_{gr} + k\tau_v u_{pr}^{(1)} + \ldots = -\frac{1}{2}kr + k\tau_v u_{pr}^{(1)} + \ldots . \tag{7a}$$

$$u_{pz} = u_{gz} + k\tau_v u_{pz}^{(1)} + \ldots = kz + k\tau_v u_{pz}^{(1)} + \ldots . \tag{7b}$$

where u_{pr} and u_{pz} are the r and z components of the particle velocity. For small slip, the particle velocity is the same as the gas velocity to leading order. The next-order correction is given by $u_{pr}^{(1)}$ and $u_{pz}^{(1)}$.

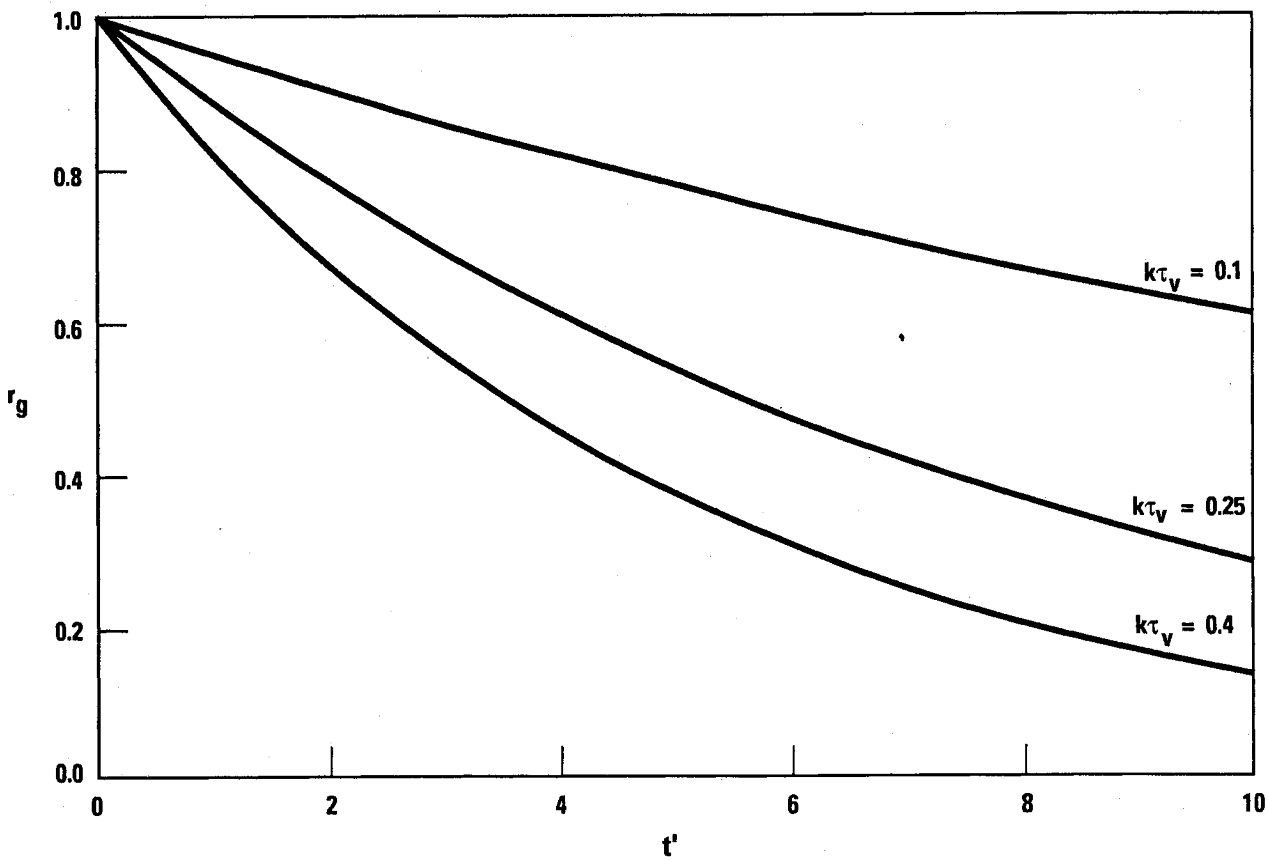

Fig. 5 *r*-component gas trajectory as a function of time.

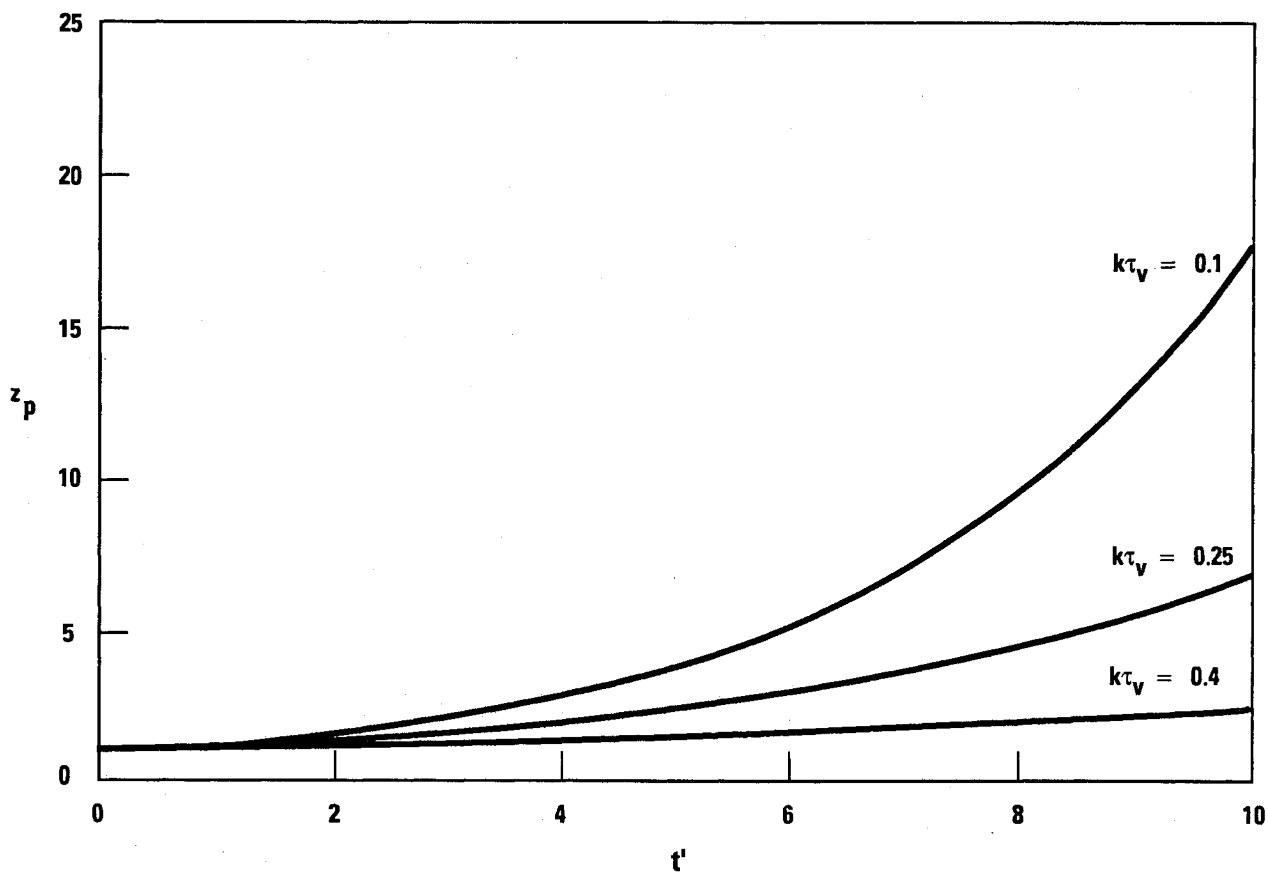

Fig. 6 *z*-component particle trajectory as a function of time.

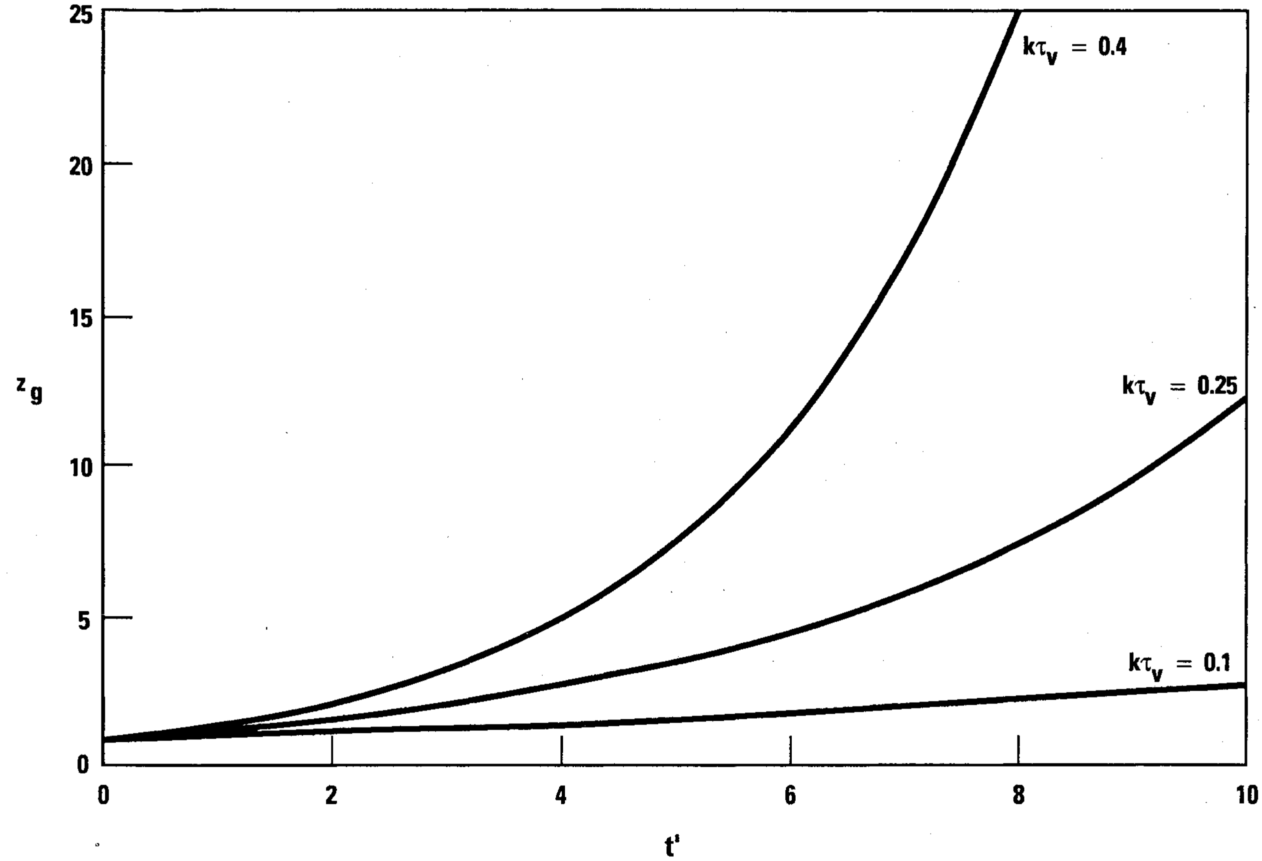

Fig. 7 *z*-**component gas trajectory as a function of time.**

The momentum equation for the particle flowfield in Eulerian formulation is

$$\boldsymbol{u}_p \cdot \nabla \boldsymbol{u}_p = \frac{1}{\tau_v}(\boldsymbol{u}_g - \boldsymbol{u}_p) \tag{8}$$

Substituting Eqs. (7a) and (7b) into Eq. (8) gives the next-order correction. The particle velocity is given by

$$u_{pr} = -\frac{1}{2}kr - \frac{1}{4}k^2\tau_v r, \qquad u_{pz} = kz - k^2\tau_v z \tag{9}$$

In general, a thin (Stokes) layer near the wall must be introduced, together with appropriate inner variables, to permit satisfaction of initial conditions imposed on the particle at the wall.

For $k\tau_v < 0.5$, the solution given by Eq. (6b) can be expressed as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{2}kr - \frac{1}{4}k^2\tau_v r + \alpha \exp\left\{\frac{1}{2}[-1 + (1 - 2k\tau_v)^{1/2}]t/\tau_v\right\} + \beta \exp\left\{\frac{1}{2}[-1 - (1 - 2k\tau_v)^{1/2}]t/\tau_v\right\} \tag{10a}$$

$$\alpha = \frac{1}{\tau_v(1 - 2k\tau_v)^{1/2}}\left\{\frac{ak\tau_v}{4}[1 + (1 - 2k\tau_v)^{1/2}] + \frac{ak^2\tau_v^2}{8}[1 + (1 - 2k\tau_v)^{1/2}] - \frac{ak\tau_v}{2}\right\} \tag{10b}$$

$$\beta = \frac{1}{\tau_v(1 - 2k\tau_v)^{1/2}}\left\{-\frac{ak\tau_v}{4}[1 + (1 - 2k\tau_v)^{1/2}] + \frac{ak\tau_v}{2} - \frac{ak^2\tau_v^2}{8}[1 - (1 - 2k\tau_v)^{1/2}]\right\} \tag{10c}$$

The quantity $\alpha = \mathcal{O}(k\tau_v)^3$ and can be neglected for $k\tau_v << 1$. The quantity $\beta = \mathcal{O}(k\tau_v)$ and is the difference between the Eulerian and Lagrangian solutions. The term associated with β in Eqs. (10) decays exponentially in time. For $\exp(-t/\tau_v) = \mathcal{O}(k^2\tau_v^2)$, the Lagrangian solution agrees with the Eulerian solution, at least to $\mathcal{O}(k\tau_v)^2$.

The Eulerian solution obtained in this section is based on the small-slip approximation and, therefore, is not valid for $k\tau_v = \mathcal{O}(1)$. In the next section, a constant-fractional-lag approximation will be proposed that is valid for larger values of $k\tau_v$.

III. Constant-Fractional-Lag Approximation for Two-Phase Flow

The Lagrangian formulation of the previous section is valid for conditions in which the particle loading is light. It can be shown[10] that, if the particle density is small compared with the gas density, the gas flowfield is not perturbed by the presence of the particles to leading order. The analytic solutions for the particle-phase quantities written in the previous section are valid only for the counterflow. More generally, the equations of motion are coupled since each component of the gas velocity is a function of both r and z; one usually must use numerical techniques in solving for the particle-phase quantities.

One interesting result from the Lagrangian formulation of particle motion is that, for particles not too close to the wall, each component of the particle velocity is almost some constant multiple of the corresponding component of the gas velocity. If a constant-fractional-lag model can be developed for at least some axisymmetric two-phase flows, such a model would be easier to implement than the Lagrangian formulation, and the solutions would be obtained in Eulerian form.

In the constant-lag model for one-dimensional compressible two-phase nozzle flow,[4,11] the ratio of the particle velocity and the gas velocity is assumed to be a constant invariant with the axial location. Various dynamic and thermodynamic quantities can be deduced easily. Kliegel termed nozzles with this property the constant-fractional-lag nozzles. The cross-sectional area of the constant-fractional-lag nozzle as a function of axial distance is available.[4]

From the results of Sec. II it appears that, more generally, the lag constant is different in different directions. The evolution equations for the particle phase can be written as

$$u_{pr}\frac{\partial u_{pr}}{\partial r} + u_{pz}\frac{\partial u_{pr}}{\partial z} = \frac{u_{gr} - u_{pr}}{\tau_v} \tag{11}$$

$$u_{pr}\frac{\partial u_{pz}}{\partial r} + u_{pz}\frac{\partial u_{pz}}{\partial z} = \frac{u_{gz} - u_{pz}}{\tau_v} \tag{12}$$

Since the gas phase is not affected by the presence of the particles, u_{gr} and u_{gz} are specified quantities in Eqs. (11) and (12). The constant-fractional-lag model is defined as

$$u_{pr} = Ju_{gr}, \qquad u_{pz} = Ku_{gz} \tag{13}$$

The definitions given in Eqs. (13) do not require any assumption concerning the particle motion. It is assumed that the coefficients J and K in Eqs. (13) are either constant or are slowly varying quantities such that their spatial variations are negligible. We seek to ascertain the conditions for which this assumption is valid. Substituting Eqs. (13) into Eqs. (11) and (12) and assuming that J and K are indeed slowly varying quantities, two equations for J and K can be obtained:

$$J^2u_{gr}\frac{\partial u_{gr}}{\partial r} + JKu_{gz}\frac{\partial u_{gr}}{\partial z} = \frac{1}{\tau_v}(1 - J)u_{gr} \tag{14}$$

$$JKu_{gr}\frac{\partial u_{gz}}{\partial r} + K^2u_{gz}\frac{\partial u_{gz}}{\partial z} = \frac{1}{\tau_v}(1 - K)u_{gz} \tag{15}$$

For the counterflow, Eqs. (14) and (15) can be solved readily. For $u_{gr} = -(1/2)\,kr$ and $u_{gz} = kz$, Eqs. (14) and (15) give

$$J = \frac{1}{k\tau_v}[1 \pm (1 - 2k\tau_v)^{1/2}] \tag{16a}$$

$$K = \frac{1}{2k\tau_v}[-1 \pm (1 + 4\,k\tau_v)^{1/2}] \tag{16b}$$

As $k\tau_v \to 0$, J and K must approach 1 since the no-slip condition has to be satisfied. This criterion leads to the determination of the signs in Eqs. (16):

$$J = \frac{1}{k\tau_v}[1 - (1 - 2\,k\tau_v)^{1/2}] \tag{17a}$$

$$K = \frac{1}{2k\tau_v}[-1 + (1 + 4k\tau_v)^{1/2}] \tag{17b}$$

It can be seen that, for the counterflow, J and K are exactly constants and, hence, the constant-lag model is valid. For small values of $k\tau_v$, one can expand J and K in terms of $k\tau_v$:

$$J \simeq 1 + \frac{1}{2}k\tau_v, \qquad K \simeq 1 - k\tau_v \tag{18}$$

The solution given by Eqs. (18) is the same as given by Eqs. (9) for small $k\tau_v$. The solution given by Eqs. (17) is more general since it does not require that $k\tau_v$ be small. Note that, if $k\tau_v > 1/2$, the expression for J in Eqs. (17) is complex and is no longer meaningful. A similar phenomenon was observed in the Lagrangian formulation for $k\tau_v > 1/2$. Physically, it corresponds to the case in which the particles reach the axis of symmetry and collide, a phenomenon omitted in the present formulation.

For a general two-phase flow, Eqs. (14) and (15) must be solved simultaneously. This leads to fourth-order-polynomial equations for J and K. A special case of practical interest occurs if u_{gr} is slowly varying in z and u_{gz} is slowly varying in r. In this case, Eqs. (14) and (15) can be solved by perturbation expansions. Let

$$J = J_o + \frac{\partial u_{gr}}{\partial z}J_1, \qquad K = K_o + \frac{\partial u_{gz}}{\partial r}K_1 \tag{19}$$

Substituting Eqs. (19) into Eqs. (14) and (15), J_o, J_1, K_o, and K_1 can be determined:

$$J_o = \frac{1}{2\tau_v(\partial u_{gr}/\partial r)}\{-1 + [1 + 4\tau_v(\partial u_{gr}/\partial r)]^{1/2}\} \tag{20a}$$

$$K_o = \frac{1}{2\tau_v(\partial u_{gr}/\partial z)}\{-1 + [1 + 4\tau_v(\partial u_{gz}/\partial z)]^{1/2}\} \tag{20b}$$

$$J_1 = \frac{-J_oK_ou_{gz}}{2J_ou_{gr}(\partial u_{gr}/\partial r) + \dfrac{1}{\tau_v}u_{gr}} \tag{20c}$$

$$K_1 = \frac{-J_oK_ou_{gr}}{2K_ou_{gz}(\partial u_{gz}/\partial z) + \dfrac{1}{\tau_v}u_{gz}} \tag{20d}$$

The solution given by Eqs. (20) is valid only if the slow variation of u_{gr} in z and u_{gz} in r is satisfied.

The solution given by Eqs. (17) for the constant-fractional-lag model is uniformly valid only if the condition of constant lag is satisfied at the wall. The constant-fractional-lag solution is an outer solution with no degrees of freedom to satisfy a boundary condition. For an arbitrary boundary condition, the constant-fractional-lag solution is valid only outside the Stokes layer, as discussed in Sec. II.

Since the constant-lag model is useful if the lag constants are slowly varying functions of positions, a formal perturbation expansion in terms of slow variables can be pursued ($R = \varepsilon r$, $Z = \varepsilon z$):

$$u_{pr} = J(R, Z)u_{gr} + \varepsilon u_{pr}^{(1)} + \dots \tag{21}$$

$$u_{pz} = K(R, Z)u_{gz} + \varepsilon u_{pz}^{(1)} + \dots \tag{22}$$

J and K satisfy Eqs. (14) and (15); $u_{pr}^{(1)}$ and $u_{pz}^{(1)}$ satisfy the following equations:

$$Ju_{gr}\frac{\partial u_{pr}^{(1)}}{\partial r} + Ku_{gz}\frac{\partial u_{pr}^{(1)}}{\partial z} + J\frac{\partial u_{gr}}{\partial r}u_{pr}^{(1)} + J\frac{\partial u_{gr}}{\partial z}u_{pz}^{(1)} + \frac{1}{\tau_v}u_{pr}^{(1)} = -J\frac{\partial J}{\partial R}u_{gr}^2 - K\frac{\partial J}{\partial Z}u_{gz}u_{gr} \tag{23}$$

$$Ju_{gr}\frac{\partial u_{pz}^{(1)}}{\partial r} + Ku_{gz}\frac{\partial u_{pz}^{(1)}}{\partial z} + K\left(u_{pr}^{(1)}\frac{\partial u_{gz}}{\partial r} + u_{pz}^{(1)}\frac{\partial u_{gz}}{\partial z}\right) + \frac{1}{\tau_v}u_{gz}^{(1)} = -J\frac{\partial K}{\partial R}u_{gr}u_{gz} - K\frac{\partial K}{\partial Z}u_{gz}^2 \tag{24}$$

Equations (23) and (24) constitute a system of two first-order partial differential equations. If the characteristics are real, then $u_{pr}^{(1)}$ and $u_{pz}^{(1)}$ can be obtained by the method of characteristics. Notice that, if $\partial u_{gr}/\partial z << 1$ and $\partial u_{gz}/\partial r << 1$, Eqs. (23) and (24) are decoupled. In applying the method of characteristics, J and K are treated as constants since they are functions of the slow variables R and Z.

IV. Conclusions

In general, if the flow cross-sectional area is slowly varying and the gas flowfield is close to the counterflow described previously, the constant-fractional-lag model should be valid. In other words, the constant-fractional-lag model should be valid over a nose-contiguous, constant-bore portion of a solid-rocket motor, and it may be even more widely useful. Extensive comparisons between the numerical solution of the Lagrangian formulation and the constant-fractional-lag model should be performed to assess the range of validity of the constant-fractional-lag model for axisymmetric two-phase flow.

Acknowledgment

The authors wish to thank George Carrier of Harvard University and James Kliegel of TRW Space & Technology Group for helpful discussions.

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